ON OPTIMAL LINEAR SMOOTHING THEORY

J. S. MEDITCH

TECHNICAL REPORT 67-105

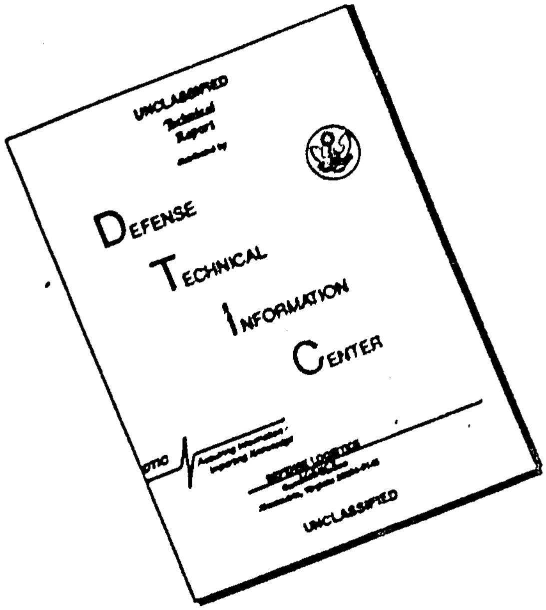
MARCH 1967

NORTHWESTERN UNIVERSITY
QUAECUMQUE SUNT VERA
1851

Information-Processing And Control SYSTEMS LABORATORY

NORTHWESTERN UNIVERSITY
EVANSTON, ILLINOIS
60201

DISCLAIMER NOTICE



THIS DOCUMENT IS BEST QUALITY AVAILABLE. THE COPY FURNISHED TO DTIC CONTAINED A SIGNIFICANT NUMBER OF PAGES WHICH DO NOT REPRODUCE LEGIBLY.

TECHNICAL REPORT 67-105

ON OPTIMAL LINEAR SMOOTHING THEORY

J. S. MEDITCH

ELECTRICAL ENGINEERING DEPARTMENT

This work was supported in part by the Joint Services Electronics Program (U.S. Army, U.S. Navy and U.S. Air Force) under Office of Naval Research Contract Number N00014-66-C0020-A03 (Identification Number NR 373-502/3-14-66 Electronics Branch).

MARCH 1967

Information-Processing And Control
SYSTEMS LABORATORY

NORTHWESTERN UNIVERSITY

EVANSTON, ILLINOIS

60201

TABLE OF CONTENTS

ON OPTIMAL LINEAR SMOOTHING THEORY

J. S. Meditch*

ABSTRACT

The algorithm for generating the smoothed estimate $\hat{x}(t/t + T)$ of the state $x(t)$ of a continuous linear system, where t is continuous time, T is a positive real constant, and $t + T$ is the time of the most recent measurement, is developed. A linear matrix differential equation whose solution gives the covariance matrix of the smoothing error $\tilde{x}(t|t + T) = x(t) - \hat{x}(t|t + T)$ is then derived. Computational aspects involved in mechanizing the algorithm are discussed in terms of the algorithm's dependence on the solution of the prediction, filtering, and fixed-point smoothing problems. The results are then discussed in terms of the classical Wiener smoothing problem.

*Department of Electrical Engineering, Northwestern University, Evanston, Illinois 60201. This work was supported in part by the U. S. Army, Navy, and Air Force in a Joint Services Electronics Program under Office of Naval Research Contract Number N00014-66-C0020-A03.

1.0 Introduction

The continuous optimal linear smoothing problem consists in determining a minimum mean-square error estimate $\hat{x}(t|\tau)$, $t_o \le t \le \tau$, of a state vector $x(t)$ based on measurements $z(\sigma)$ given over the entire interval $t_o \le \sigma \le \tau$ where

$$\dot{x} = F(t)\ x + w(t) \tag{1}$$

$$z(t) = H(t)\ x(t) + v(t) \tag{2}$$

In Eqs. (1) and (2), x is an n-vector, the state; z is an m-vector, the measurement; w is an n-vector, the disturbance; v is an m-vector, the measurement error; F and H are continuous n x n and m x n matrices, respectively; t denotes time; and the dot denotes the time derivative. We assume that $w(t)$ and $v(t)$ are independent Gaussian white noise processes with identically zero means and covariance matrices

$$E[\,w(t)\ w'(\sigma)\,] = Q(t)\ \delta(t - \sigma)$$

$$E[\,v(t)\ v'(\sigma)\,] = R(t)\ \delta(t - \sigma)$$

for all t and σ where E denotes the expected value, the prime denotes the transpose, δ is the Dirac delta function, $Q(t)$ is a symmetric n x n positive semidefinite matrix, and $R(t)$ is a symmetric m x m positive definite matrix.

We assume further that the initial time t_o is fixed, and that $x(t_o)$ is a zero mean Gaussian random n-vector which is independent of $u(t)$ and $v(t)$ for all t and whose covariance matrix $P(t_o) = E[\,x(t_o)\ x'(t_o)\,]$ is a symmetric n x n positive semidefinite matrix.

We define the smoothing error by the relation

$$\tilde{x}(t|\tau) = x(t) - \hat{x}(t|\tau)$$

and the mean-square smoothing error by

$$S = E[\,\tilde{x}'(t|\tau)\ \tilde{x}(t|\tau)\,]$$

where $t \le \tau$. We call an estimate $\hat{x}(t|\tau)$ that minimizes S an optimal smoothed estimate.

We note that there are basically three separate cases to be considered in the

smoothing problem:

1. <u>Fixed-Interval Smoothing</u>. In this case, the interval $[t_o, \tau]$ over which the measurements are given is fixed, and we obtain an optimal smoothed estimate of x for all $t \in [t_o, \tau]$. This case has been treated by Bryson and Frazier[1], and Rauch, Tung, and Striebel[2]. The results are of particular significance in post-experimental data analysis where one needs to obtain a "refined" estimate of the state vector of a physical system over the system's entire operating time.

2. <u>Fixed-Point Smoothing</u>. If an optimal smoothed estimate of x is desired at only one value of $t \in [t_o, \tau]$, we call $\hat{x}(t|\tau)$ a fixed-point smoothed estimate. Such an estimate can be obtained using fixed-interval smoothing, but the procedure is computationally inefficient for this purpose[3,4]. Moreover, if the terminal measurement time τ is not specified <u>a priori</u>, as might be the case in an "on-line" smoothing problem, fixed-interval smoothing is not applicable here. The algorithm for optimal fixed-point smoothing for continuous linear systems which begins with the optimal filtered estimate[5] of x at the fixed time t and "updates" this estimate recursively as more measurement data become available has been developed by Meditch[4]. The procedure is applicable to both post-experimental data analysis and "on-line" data processing problems where one requires a smoothed estimate of a physical system's state at some critical time during the system's operation.

3. <u>Fixed-Lag Smoothing</u>. Now suppose that T is replaced by t + T where T = constant > 0 and t is variable with $t \geq t_o$. Then, we see that $\hat{x}(t|t + T)$ is a "running" smoothed estimate which "lags" behind the time of the most recent measurement by a fixed amount T. For obvious reasons, we call $\hat{x}(t|t + T)$ a fixed-lag smoothed estimate. Such estimates are primarily of interest in communication and telemetry systems where one wishes an "on-line" smoothed estimate of the state of the message or data transmitted. The intuitive justification for the lag is that we would expect less mean-square error in the smoothed estimate than in the predicted or filtered estimates $\hat{x}(t|\sigma)$, $t > \sigma$ and $t = \sigma$, respectively.

In this paper, we shall develop the algorithm for optimal fixed-lag smoothing for continuous linear systems of the type described by Eqs. (1) and (2). We shall

also derive the matrix differential equation whose solution gives the covariance matrix of the fixed-lag smoothing error

$$\tilde{x}(t|t + T) = x(t) - \hat{x}(t|t + T)$$

Our approach consists in considering the limiting case of the fixed-lag smoothing solution for discrete linear systems wherein the time between measurements is made arbitrarily small. The equations for discrete optimal fixed-lag smoothing are well-known(3), and will be used as the starting point in our work. The limiting process that we shall utilize is due to Kalman(6).

2.0 Optimal Fixed-Lag Discrete Linear Smoothing

We begin by considering the discrete system analog of Eqs. (1) and (2) which can be expressed in the form

$$x(k + 1) = \Phi(k + 1,k)\ x(k) + w(k) \tag{3}$$

$$z(k + 1) = H(k + 1)\ x(k + 1) + v(k + 1) \tag{4}$$

where v, w, x, z, and H are of the same dimensions as they were in Eqs. (1) and (2); Φ is an nxn matrix, the state transition matrix; and k = 0,1,..., is the discrete time index. We assume that w and v are independent Gaussian white sequences with identically zero means and covariance matrices

$$E\left[w(j)\ w'(k)\right] = Q(k)\ \delta_{jk}$$

and

$$E\left[v(j)\ v'(k)\right] = R(k)\ \delta_{jk}$$

for all j and k where δ_{jk} is the Kronecker delta, Q(k) is a symmetrix n x n positive semidefinite matrix, and R(k) is a symmetric m x m positive definite matrix.

We further assume that the initial state x(0) is a zero mean Gaussian random n-vector which is independent of w(k) and v(k) for all k and whose covariance matrix P(0) = E [x(0) x'(0)] is n x n symmetric, and positive semidefinite.

We let N be some fixed positive integer and denote a fixed-lag smoothed estimate of x(k), given measurements up to and including the one at time k + N, by

$\hat{x}(k|k+N)$ where $k = 0,1,\ldots$. We define the corresponding smoothing error and its mean-square value by the relations

$$\tilde{x}(k|k+N) = x(k) - \hat{x}(k|k+N)$$

and

$$S = E[\tilde{x}'(k|k+N)\,\tilde{x}(k|k+N)]$$

respectively. We call an estimate $\hat{x}(k|k+N)$ that minimizes S the optimal fixed-lag smoothed estimate. It has been shown[3] that this estimate is given recursively by the system of n first-order difference equations

$$\begin{aligned}\hat{x}(k+1|k+1+N) = {} & \Phi(k+1,k)\,\hat{x}(k|k+N) \\ & + C(k+N,k+1)\,K(k+1+N) \\ & \quad \cdot [z(k+1+N) - H(k+1+N)\,\hat{x}(k+1+N|k+N)] \\ & + Q(k)\,\Phi'(k,k+1)\,P^{-1}(k) \\ & \quad \cdot [\hat{x}(k|k+N) - \hat{x}(k)]\end{aligned} \tag{5}$$

for $k = 0,1,\ldots,$ where

$$C(k+N,k+1) = \prod_{i=k+1}^{k+N} J(i) \tag{6}$$

$$J(i) = P(i)\,\Phi'(i+1,i)\,M^{-1}(i+1) \tag{7}$$

$$K(k+1+N) = P(k+1+N)\,H'(k+1+N)\,R^{-1}(k+1+N) \tag{8}$$

$[\;]^{-1}$ denotes the matrix inverse, and $\hat{x}(k)$ and $\hat{x}(k+1+N|k+N)$ are the filtered and predicted estimates of $x(k)$ and $x(k+1+N)$, respectively. In addition, the $n \times n$ matrices P and M are the covariance matrices of the filtering and prediction errors

$$\tilde{x}(j+1) = x(j+1) - \hat{x}(j+1)$$

and

$$\tilde{x}(j+1|j) = x(j+1) - \hat{x}(j+1|j)$$

respectively.

The filtered and predicted estimates along with their corresponding error covariance matrices are governed by the set of relations(6,7)

$$\hat{x}(j+1|j) = \Phi(j+1,j)\ \hat{x}(j) \tag{9}$$

$$\hat{x}(j+1) = \hat{x}(j+1|j) + K(j+1)\ [z(j+1) - H(j+1)\ \hat{x}(j+1|j)] \tag{10}$$

$$K(j+1) = M(j+1)\ H'(j+1)\ [H(j+1)\ M(j+1)\ H'(j+1) + R(j+1)]^{-1} \tag{11}$$

$$M(j+1) = \Phi(j+1,j)\ P(j)\ \Phi'(j+1,j) + Q(j) \tag{12}$$

$$P(j+1) = [I - K(j+1)\ H(j+1)]\ M(j+1) \tag{13}$$

for $j = 0,1,\ldots$, where $\hat{x}(0) = 0$, $P(0) = E[\,x(0)\ x'(0)\,]$, and I is the $n \times n$ identity matrix.

The $n \times m$ matrix K which is given by either Eq.(8) or Eq.(11) and which also appears in Eq. (5) is called the optimal filter gain.

In order to initiate fixed-lag smoothing, we note that at $k = 0$, $\hat{x}(N+1|N)$, $\hat{x}(0)$, and $\hat{x}(0|N)$ must be input to Eq. (5). The first two of these follow directly from the results for prediction and filtering. However, $\hat{x}(0|N)$ must be obtained from the fixed-point smoothing filter by starting with $\hat{x}(0) = 0$ and processing the measurements at $k = 1,\ldots,N$. The equations necessary to do this are given elsewhere(3,8) for the discrete case. They are of no consequence here since we are concerned with the continuous case. After we have developed the algorithm for continuous fixed-lag smoothing, we shall show in detail what procedure must be followed to obtain the appropriate initial conditions. In any event, it is clear that fixed-lag smoothing depends upon inputs from both the predictor-filter and the fixed-point smoothing filter.

The covariance matrix of the fixed-lag discrete optimal linear smoothing error which was defined earlier is given by the first-order $n \times n$ matrix difference equation

$$P(k+1|k+1+N) = M(k+1) - C(k+N,k+1)\, K(k+1+N)\, H(k+1+N)\, M(k+1+N)\, C'(k+N,k+1)$$
$$- J^{-1}(k)[\, P(k) - P(k|k+N)\,]\, J^{-1'}(k) \tag{14}$$

for k = 0,1,..., where

$$P(k|k + N) = E\,[\tilde{x}(k|k + N)\,\tilde{x}'(k|k + N)\,]$$

and all of the other terms were defined previously.

The initial condition on Eq. (14) is $P(0|N)$ which must be obtained from the covariance equation for fixed-point smoothing[3,8]. We shall present the equation whose solution will give the initial condition for the continuous version of Eq. (14) in Section 4.0.

3.0 Optimal Fixed-Lag Continuous Linear Smoothing

Let us assume that the system of Eqs. (3) and (4) has been developed by discretizing the system of Eqs. (1) and (2). The corresponding fixed-lag smoothing filter is then defined by Eqs. (5) through (8). We now consider the limiting behavior of this latter set of equations as the time between measurements is made arbitrarily small.

We let the discrete time instants k and k + 1 be denoted by t and $t + \Delta t$, respectively, where $\Delta t > 0$. We let the time interval N be denoted by T = constant > 0, and, as a result, see that k + N and k + 1 + N become t + T and $t + \Delta t + T$, respectively. In considering the limiting case, the covariance matrices Q(k) and R(k) must be replaced by $Q(t)/\Delta t$ and $R(t)/\Delta t$, respectively. This is necessary in order to obtain a physically meaningful description of the Gaussian white noise as the limit of the Gaussian white sequence. In addition, a factor of Δt arises in the disturbance term so that w(k) is replaced by $w(t)\Delta t$. As a result, Q(k) is replaced by $Q(t)\Delta t$ in all covariance relations involving Q(k). The details of the justification for this procedure are given elsewhere[2,6,9], and will not be repeated here.

Making these substitutions into Eqs. (5) and (8), we have

$$\hat{x}(t+\Delta t|t+\Delta t+T) = \Phi(t+\Delta t,t)\, \hat{x}(t|t+T) + C(t+T,\, t+\Delta t)\, K(t+\Delta t+T)$$
$$\cdot [\, z(t+\Delta t+T) - H(t+\Delta t+T)\, \hat{x}(t+\Delta t+T|t+T)\,]$$
$$+ Q(t)\, \Phi'(t,t+\Delta t)\, P^{-1}(t)[\, \hat{x}(t|t+T) - \hat{x}(t)\,]\, \Delta t \tag{15}$$

and

$$K(t+\Delta t+T) = P(t+\Delta t+T)\ H'(t+\Delta t+T)\ R^{-1}(t+\Delta t+T)\,\Delta t \tag{16}$$

respectively.

Since the state transition matrix satisfies the relations

$$\dot{\Phi}(t,\tau) = F(t)\ \Phi(t,\tau) \qquad \text{and} \qquad \Phi(\tau,\tau) = I \qquad \text{for all } \tau,$$

we see that $\Phi(t + \Delta t,t)$ and $\Phi'(t,t, +\Delta t)$ can be expanded in the Taylor series'

$$\Phi(t + \Delta t,t) = I + F(t)\,\Delta t + 0(\Delta t^2) \tag{17}$$

and

$$\Phi'(t,t + \Delta t) = I - F'(t)\,\Delta t + 0(\Delta t^2) \tag{18}$$

respectively, where $0(\Delta t^2)$ denotes terms of order $(\Delta t)^2$.

Substituting Eqs. (16), (17), and (18) into Eq. (15), and rearranging terms, we obtain

$$\begin{aligned}\hat{x}(t+\Delta t|t+\Delta t+T) - \hat{x}(t|t+T) &= F(t)\ \hat{x}(t|t+T)\Delta t + C(t+T,t+\Delta t)\ P(t+\Delta t+T)\\ &\cdot H(t+\Delta t+T)\ R^{-1}(t+\Delta t+T)[\ z(t+\Delta t+T) - H(t+\Delta t+T)\ \hat{x}(t+\Delta t+T|t+T)]\Delta t\\ &+ Q(t)\ P^{-1}(t)\ [\hat{x}(t|t+T) - \hat{x}(t)]\ \Delta t + 0(\Delta t^2)\end{aligned}$$

Dividing through by Δt and taking $\lim \Delta t \to 0$, we then have that

$$\begin{aligned}\dot{\hat{x}}(t|t\Delta+ T) &= F(t)\ \hat{x}(t|t + T) + \lim_{\Delta t \to 0} \Big\{C(t + T,t + \Delta t)\ P(t + \Delta t + T)\\ &\cdot H'(t+\Delta t+T)\ R^{-1}(t+\Delta t+T)[\ z(t+\Delta t+T) - H(t+\Delta t+T)\ \hat{x}(t+\Delta t+T|t+T)]\Big\}\\ &+ Q(t)\ P^{-1}(t)[\ \hat{x}(t|t + T) - \hat{x}(t)]\end{aligned} \tag{19}$$

where it remains for us to evaluate the second term on the right-hand side.

From Eq. (6), it is seen that

$$C(t+T,t) = J(t)\, C(t+T,t+\Delta t)$$

which can also be written

$$C(t+T,t+\Delta t) = J^{-1}(t)\, C(t+T,t) \tag{20}$$

From Eq. (7),

$$J^{-1}(t) = M(t+\Delta t)\, \Phi'(t,t+\Delta t)\, P^{-1}(t) \tag{21}$$

Making the appropriate substitutions into Eq. (12), we obtain the result

$$\begin{aligned} M(t+\Delta t) &= \Phi(t+\Delta t,t)\, P(t)\, \Phi'(t+\Delta t,t) + Q(t)\,\Delta t \\ &= [I + F(t)\Delta t + 0(\Delta t^2)]\, P(t)\, [I + F(t)\Delta t + 0(\Delta t^2)]' + Q(t)\Delta t \\ &= P(t) + [F(t)\, P(t) + P(t)\, F'(t) + Q(t)]\,\Delta t + 0(\Delta t^2) \\ &= P(t) + 0(\Delta t) \end{aligned} \tag{22}$$

where $0(\Delta t)$ denotes terms of order Δt.

Substituting Eqs. (22) and (18) into Eq. (21) and simplifying the result, we see that

$$\begin{aligned} J^{-1}(t) &= [P(t) + 0(\Delta t)]\, [I - F'(t)\Delta t + 0(\Delta t^2)]\, P^{-1}(t) \\ &= I + 0(\Delta t) \end{aligned}$$

Hence, Eq. (20) can be written

$$C(t+T,t+\Delta t) = C(t+T,t) + 0(\Delta t)$$

from which it immediately follows that

$$\lim_{\Delta t \to 0} C(t+T,t+\Delta t) = C(t+T,t) \tag{23}$$

From the nature of the matrices P, H, and K^{-1}, it also follows that

$$\lim_{\Delta t \to 0} P(t+\Delta t+T)\ H'(t+\Delta t+T)\ R^{-1}(t+\Delta t+T) = P(t+T)\ H'(t+T)\ R^{-1}(t+T) \tag{24}$$

From Eqs. (9) and (7), we see that

$$\begin{aligned}\hat{x}(t + \Delta t + T | t + T) &= \Phi(t + \Delta t + T, t + T)\ \hat{x}(t + T) \\ &= [I + F(t)\Delta t + 0(\Delta t^2)]\ \hat{x}(t + T) \\ &= \hat{x}(t + T) + 0(\Delta t)\end{aligned}$$

Hence,

$$\lim_{\Delta t \to 0} [\ z(t+\Delta t+T) - H(t+\Delta t+T)\ \hat{x}(t+\Delta t+T | t+T)\ ] = z(t+T) - H(t+T)\ \hat{x}(t+T) \tag{25}$$

As a consequence of Eqs.(23,(24),and (25),the second term on the right-hand side of Eq. (19) can be evaluated as the product of the three limits in these equations, and we are led to the result

$$\begin{aligned}\dot{\hat{x}}(t | t+T) = F(t)\ \hat{x}(t | t+T) &+ C(t+T,t)\ P(t+T)\ H'(t+T)\ R^{-1}(t+T) \\ &\cdot[z(t+T) - H(t+T)\ \hat{x}(t+T)] + Q(t)\ P^{-1}(t)\ [\hat{x}(t | t+T) - \hat{x}(t)]\end{aligned} \tag{26}$$

which specifies the optimal fixed-lag continuous linear smoothing filter.

Since

$$K(t + T) = P(t + T)\ H'(t + T)\ R^{-1}(t + T)$$

is the n x m gain matrix of the optimal continuous linear filter[5,6],we can also write Eq. (26) as

$$\begin{aligned}\dot{\hat{x}}(t | t+T) = F(t)\ \hat{x}(t | t+T) &+ C(t+T,t)\ K(t+T)\ [z(t+T) - H(t+T)\ \hat{x}(t+T)] \\ &+ Q(t)\ P^{-1}(t)\ [\hat{x}(t | t+T) - \hat{x}(t)]\end{aligned} \tag{27}$$

where $t \geq t_o$.

We note that mechanization of Eq. (27) requires the following input data:

A. The system matrix $F(t)$ and the system disturbance covariance matrix $Q(t)$.

B. The "gain times residual" term $K(t + T)\,[z(t + T) - H(t + T)\,\hat{x}(t + T)]$.

C. The time history of the $n \times n$ smoothing filter gain matrix $C(t + T,t)$.

D. The optimal filtered estimate $\hat{x}(t)$ and the inverse of its error covariance matrix, i.e., $P^{-1}(t)$

E. The initial condition $\hat{x}(t_o|t_o + T)$

Item A is generally given in the problem specification. Items B and D follow directly from the optimal filter results(5,6) which we repeat here for convenience:

$$\dot{\hat{x}} = F(\sigma)\,\hat{x} + K(\sigma)\,[\,z(\sigma) - H(\sigma)\,\hat{x}\,] \tag{28}$$

$$K(\sigma) = P(\sigma)\,H'(\sigma)\,R^{-1}(\sigma) \tag{29}$$

$$\dot{P} = F(\sigma)\,P + PF'(\sigma) - PH'(\sigma)\,R^{-1}(\sigma)\,H(\sigma)\,P + Q(\sigma) \tag{30}$$

where $\sigma \geq t_o$, $\hat{x}(t_o)=0, P(t_o) = E\,[x(t_o)\,x'(t_o)]$, and the dot denotes the derivative with respect to σ. In this connection, we observe that the filter of Eq. (28) must process the measurements in the interval $[t_o, t_o + T\,]$ before smoothing is initiated since the term $K(t_o + T)]\,z(t_o + T) - H(t_o + T)\,\hat{x}(t_o + T)]$ is required in Eq. (27) to begin smoothing. As a result, we have two time scales here: the filter time scale σ and the smoothing filter time scale t where $t = \sigma - T$, i.e., the smoothing filter must, of necessity, "lag" the filter of Eq. (28) by T units of time in executing its data processing.

Because of the relationship between the two time scales, we see that $P^{-1}(t)$ and $\hat{x}(t)$ which are required in Eq. (27) can be obtained directly from $P(\sigma)$ and $x(\sigma)$ which must be computed first in order to mechanize Eq. (28) by introducing a time delay of T units into the latter two quantities.

We summarize our discussion here in the form of the block diagram shown below in Fig. 1.

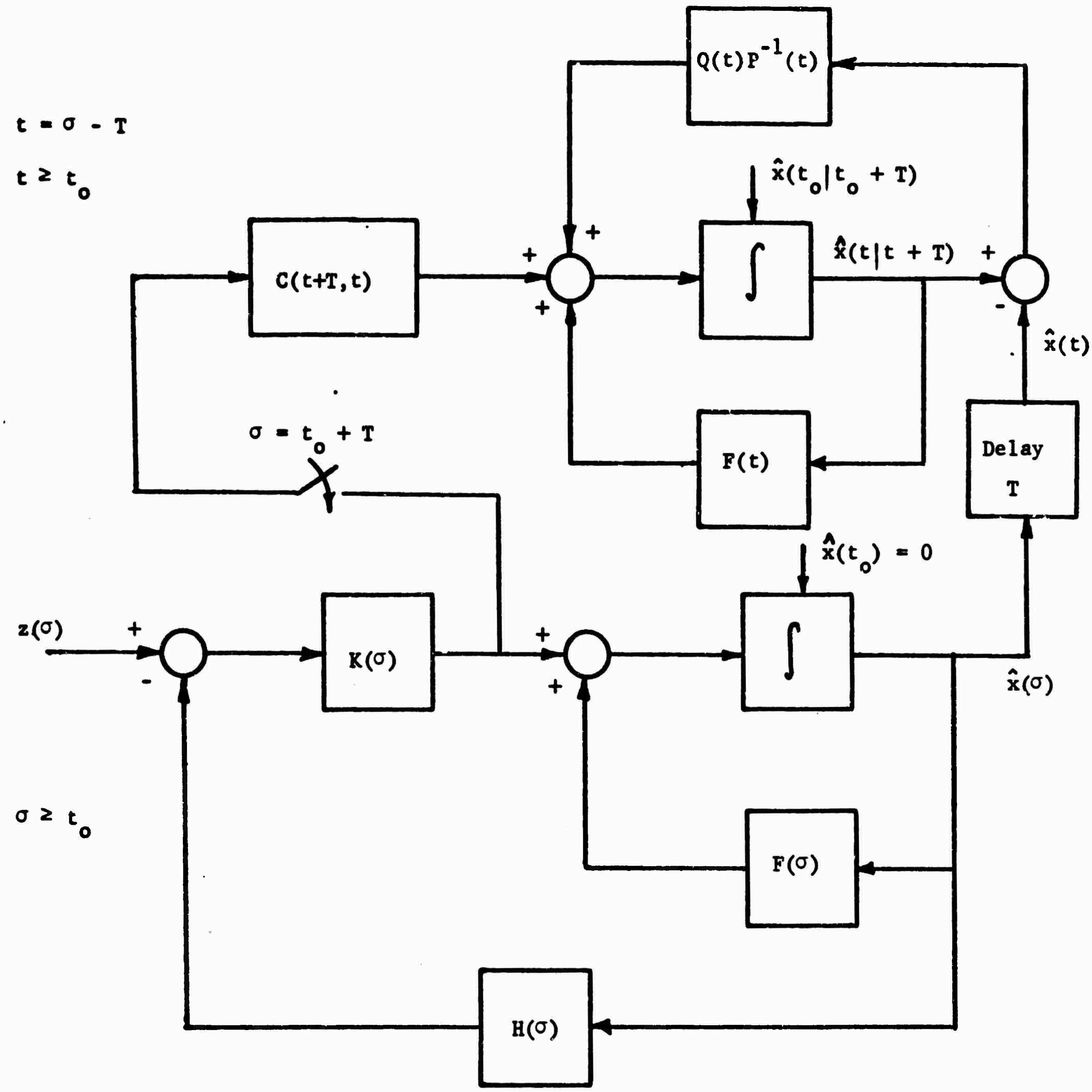


Fig. 1

Smoothing filter block diagram.

We turn now to consideration of items C and E. In particular, the initial condition $\hat{x}(t_o|t_o + T)$ can be obtained by utilizing optimal fixed-point continuous linear smoothing[4] to process the measurements in the interval $[t_o, t_o + T]$. The required algorithm is[4]

$$\dot{\hat{x}}(t_o|\sigma) = B(\sigma, t_o)\, K(\sigma)\, [z(\sigma) - H(\sigma)\, \hat{x}(\sigma)] \tag{31}$$

where $\sigma \geq t_o$, $\hat{x}(t_o|t_o) = 0$, and the n x n gain matrix satisfies the matrix differential equation

$$\dot{B}(\sigma,t_o) = -B(\sigma,t_o)[F(\sigma) + Q(\sigma)P^{-1}(\sigma)] \tag{32}$$

where $B(t_o,t_o) = I$.

The term $K(\sigma)[z(\sigma) - H(\sigma)x(\sigma)]$ is obtained directly from the optimal filter defined by Eqs. (28) through (30), and the fixed-point smoothing filter can be operated on the same time scale, i.e., simultaneously, with the filter of Eq. (28)[4].

At $\sigma = t_o + T$, fixed-point smoothing is terminated, and the output $\hat{x}(t_o|t_o + T)$ of this smoothing procedure is input as the initial condition for fixed-lag smoothing (see Eq. (27) and Fig. 1). We note, of course, that the fixed-lag smoothing filter has been inoperative during the interval $t_o \leq \sigma \leq t_o + T$ while "waiting" for the required initial condition.

We conclude this section by developing the matrix differential equation whose solution is the fixed-lag smoothing filter gain $C(t + T,t)$. From Eq. (6), we see that

$$C(k+1+N,k+1) = \prod_{i=k+1}^{k+1+N} J(i)$$

$$= J^{-1}(k)\left[\prod_{i=k}^{k+N} J(i)\right] J(k+1+N)$$

$$= J^{-1}(k)\,C(k+N,k)\,J(k+1+N)$$

which we choose to express in the form

$$C(k+1+N,k+1)\,J^{-1}(k+1+N) = J^{-1}(k)\,C(k+N,k) \tag{33}$$

Replacing k by t, k + 1 by $t + \Delta t$, k + N by t + T, and k + 1 + N by $t + \Delta t + T$ in Eq. (33), we obtain

$$C(t + \Delta t + T, t + \Delta t)\ J^{-1}(t + \Delta t + T) = J^{-1}(t)\ C(t + T, t) \tag{34}$$

From Eq. (21), we recall that

$$J^{-1}(t) = M(t + \Delta t)\ \Phi'(t, t + \Delta t)\ P^{-1}(t) \tag{21}$$

From the first line in Eq. (22), and Eqs. (17) and (18), we see that

$$\begin{aligned} M(t + \Delta t)\ \Phi'(t, t + \Delta t) &= \Phi(t + \Delta t, t)\ P(t) + Q(t)\ \Phi'(t, t + \Delta t)\ \Delta t \\ &= [I + F(t)\Delta t + 0(\Delta t^2)]\ P(t) \\ &\quad + Q(t)\ [I - F'(t)\Delta t + 0(\Delta t^2)]\Delta t \\ &= P(t) + [F(t)P(t) + Q(t)]\Delta t + 0(\Delta t^2) \end{aligned}$$

Postmultiplying this result by $P^{-1}(t)$, we obtain the result

$$J^{-1}(t) = I + [F(t) + Q(t)\ P^{-1}(t)]\Delta t + 0(\Delta t^2) \tag{35}$$

In an identical manner, we also have

$$J^{-1}(t+\Delta t+T) = I + [F(t+\Delta t+T) + Q(t+\Delta t+T)\ P^{-1}(t+\Delta t+T)]\Delta t + 0(\Delta t^2) \tag{36}$$

Substituting Eqs. (35) and (36) into Eq. (34) and rearranging the result gives us

$$\begin{aligned} C(t+\Delta t+T, t+\Delta t) - C(t+T, t) &= [F(t)+Q(t)P^{-1}(t)]C(t+T,t)\Delta t \\ &\quad - C(t+\Delta t+T, t+\Delta t)[F(t+\Delta t+T)+Q(t+\Delta t+T)P^{-1}(t+\Delta t+T)]\ \Delta t \\ &\quad + 0(\Delta t^2) \end{aligned}$$

Dividing through by Δt and taking $\lim \Delta t \to 0$, we then have

$$\dot{C}(t+T,t) = [F(t)+Q(t)P^{-1}(t)]\ C(t+T,t) - C(t+T,t)[F(t+T)+Q(t+T)P^{-1}(t+T)] \tag{37}$$

which is the desired result. The initial condition for Eq. (37) is $C(t_o + T, t_o) =$

$B(t_o + T, t_o)$ which is simply the solution of Eq. (32) evaluated at $\sigma = t_o + T$. In this case, we observe that computation of $C(t + T, t)$ where $t \geq t_o$ cannot be initiated until both Eqs. (30) and (37) have been solved over the interval $t_o \leq \sigma \leq t_o + T$ to obtain $P(t_o + T)$ and $B(t_o + T, t_o)$, respectively. We note, of course, that $P(\sigma)$ is also required for $\sigma > t_o + T$. Finally, although the gain matrix $C(t + T, t)$ can be computed a priori, i.e., before fixed-lag smoothing is initiated, by solving Eqs. (30), (32), and (37), the fact remains that the smoothing filter of Eq. (27) cannot begin functioning until $\hat{x}(t_o | t_o + T)$ and the term $K(t_o + T)\,[z(t_o + T) - H(t_o + T) \cdot \hat{x}(t_o + T)]$ are determined by fixed-point smoothing and filtering, respectively, over the interval $t_o \leq \sigma \leq t_o + T$.

4.0 Optimal Fixed-Lag Continuous Linear Smoothing Error Covariance

Replacing k by t, k + 1 by t + Δt, k + N by t + T, and k + 1 + N by t + Δt + T in Eq. (14), and substituting into this result from Eq. (16), we have

$$P(t+\Delta t | t+\Delta t+T) = M(t+\Delta t) - C(t+T, t+\Delta t) P(t+\Delta t+T) H'(t+\Delta t+T) R^{-1}(t+\Delta t+T)$$
$$H(t+\Delta t+T) M(t+\Delta t+T) C'(t+T, t+\Delta t)\ \Delta t$$
$$- J^{-1}(t)\ [P(t) - P(t|t+T)]\ J^{-1'}(t) \tag{38}$$

For the present, we focus our attention on the first and third terms of Eq. (38).

We recall from Eq. (22) that

$$M(t + \Delta t) = P(t) + [F(t)\ P(t) + P(t)\ F'(t) + Q(t)]\Delta t + 0(\Delta t^2) \tag{22}$$

Utilizing the expression for $J^{-1}(t)$ as given in Eq. (35), we observe that

$$J^{-1}(t)\ P(t)\ J^{-1'}(t) = \left\{ I + [F(t)+Q(t)P^{-1}(t)]\Delta t + 0(\Delta t^2) \right\} P(t)$$
$$\cdot \left\{ I + [F'(t)+P^{-1}(t)Q(t)]\ \Delta t + 0(\Delta t^2) \right\}$$
$$= \left\{ P(t) + [F(t)P(t)+Q(t)]\ \Delta t + 0(\Delta t^2) \right\}$$
$$\cdot \left\{ I + [F'(t)+P^{-1}(t)Q(t)]\ \Delta t + 0(\Delta t^2) \right\}$$
$$= P(t) + [F(t)P(t)+Q(t)]\ \Delta t + [P(t)F'(t)+Q(t)]\ \Delta t + 0(\Delta t^2)$$

$$= P(t) + [F(t)P(t)+P(t)F'(t)+2Q(t)]\ \Delta t + 0(\Delta t^2) \tag{39}$$

Similarly, we have that

$$\begin{aligned}J^{-1}(t)P(t|t+T)J^{-1'}(t) &= \{P(t|t+T)+[F(t)+Q(t)P^{-1}(t)]\ P(t|t+T)\Delta t + 0(\Delta t^2)\} \\ &\quad \cdot \{I + [F'(t)+P^{-1}(t)Q(t)]\ \Delta t + 0(\Delta t^2)\} \\ &= P(t|t+T)+[F(t)+Q(t)P^{-1}(t)]\ P(t|t+T)\ \Delta t \\ &\quad + P(t|t+T)\ [F(t)+Q(t)P^{-1}(t)]'\ \Delta t + 0(\Delta t^2)\end{aligned} \tag{40}$$

Combining Eqs. (22), (39), and (40), we see that

$$\begin{aligned}M(t+ t)-J^{-1}(t)[P(t)-P(t|t+T)]J^{-1'}(t) &= P(t|t+T)+[F(t)+Q(t)P^{-1}(t)]\ P(t|t+T)\Delta t \\ &\quad + P(t|t+T)\ [F(t)+Q(t)P^{-1}(t)]'\Delta t-Q(t)\Delta t \\ &\quad + 0(\Delta t^2)\end{aligned} \tag{41}$$

Substituting Eq. (41) into Eq. (38) and rearranging terms, we have

$$\begin{aligned}P(t+\Delta t|t+\Delta t+T)-P(t|t+T) &= [F(t)+Q(t)P^{-1}(t)]P(t|t+T)\Delta t+P(t|t+T)[F(t)+Q(t)P^{-1}(t)]'\Delta t \\ &\quad - C(t+T,t+\Delta t)P(t+\Delta t+T)H'(t+\Delta t+T)R^{-1}(t+\Delta t+T)H(t+\Delta t+T) \\ &\quad \cdot M(t+\Delta t+T)C'(t+T,t+\Delta t)\ \Delta t - Q(t)\ \Delta t + 0(\Delta t^2)\end{aligned}$$

Dividing through by Δt, taking $\lim \Delta t \to 0$, utilizing the results in Eqs. (23) and (24), and noting from Eq. (22) that

$$\lim_{\Delta t \to 0} M(t + \Delta t + T) = P(t + T)$$

we obtain

$$\begin{aligned}\dot{P}(t|t+T) &= [F(t)+Q(t)P^{-1}(t)]\ P(t|t+T) + P(t|t+T)\ [F(t)+Q(t)P^{-1}(t)]' \\ &\quad - C(t+T,t)P(t+T)H'(t+T)R^{-1}(t+T)H(t+T)P(t+T)C'(t+T,t) - Q(t)\end{aligned} \tag{42}$$

which is the result sought.

The n x n matrix $C(t + T,t)$ in Eq.(42) is the solution of Eq.(37). The covariance matrices $P^{-1}(t)$ and $P(t + T)$ are obtained from the solution of the filter error covariance relation, Eq. (30), as described previously. Finally, the initial condition for Eq. (42) is $P(t_o|t_o + T)$ which can be obtained by solving the fixed-point smoothing error covariance equation[4]

$$\dot{P}(t_o|\sigma) = - B(\sigma,t_o)P(\sigma)H'(\sigma)R^{-1}(\sigma)H(\sigma)P(\sigma)B'(\sigma,t_o) \tag{43}$$

over the interval $t_o \leq \sigma \leq t_o + T$ where $B(\sigma,t_o)$ is the solution of Eq. (32) and the initial condition for Eq. (43) is $P(t_o)$.

Solution of Eq. (42) then gives the covariance matrix of the optimal fixed-lag continuous linear smoothing error $\tilde{x}(t|t + T) = x(t) - \hat{x}(t|t + T)$ and it follows that

$$E[\tilde{x}'(t|t + T)\ \tilde{x}(t|t + T)] = \text{trace}\ P(t|t + T)$$

Finally, by noting the definition of the optimal filter gain matrix K, we see that Eq. (42) can also be written as

$$\dot{P}(t|t+T) = [F(t)+Q(t)P^{-1}(t)]\,P(t|t+T) + P(t|t+T)\,[F(t)+Q(t)P^{-1}(t)]'$$
$$- C(t+T,t)\ K(t+T)\ H(t+T)\ P(t+T)\ C'(t+T,t) - Q(t) \tag{43}$$

5.0 Discussion of Results

For convenience of reference in the discussion to follow, let us summarize the results for optimal fixed-lag continuous linear smoothing. The smoothing filter equation is

$$\dot{\hat{x}}(t|t+T) = F(t)\ \hat{x}(t|t+T) + C(t+T,t)\ K(t+T)\ [z(t+T) - H(t+T)\ \hat{x}(t+T)]$$
$$+ Q(t)\ P^{-1}(t)\ [\hat{x}(t/t+T) - \hat{x}(t)] \tag{27}$$

where

$$K(t+T) = P(t+T)\ H'(t+T)\ R^{-1}(t+T)$$

and $C(t,t+T)$ is the solution of the n x n matrix differential equation

$$\dot{C}(t+T,t) = F(t)+Q(t)P^{-1}(t) \quad C(t+T,t) - C(t+T,t) \quad F(t+T)+Q(t+T)P^{-1}(t+T) \qquad (37)$$

In these three equations $t \geq t_o$, and the initial conditions $\hat{x}(t_o | t_o + T)$ and $C(t_o + T, t_o)$ are obtained from the solution of the optimal fixed-lag continuous linear smoothing problem over the interval $[t_o, t_o + T]$. The n x m matrix $K(t + T)$ is the gain matrix for optimal continuous linear filtering. The matrix $C(t + T,t)$ is termed the optimal smoothing filter gain.

The fixed-lag smoothing error covariance matrix equation is

$$\dot{P}(t| t+T) = [F(t)+Q(t)P^{-1}(t)] \, P(t| t+T) + P(t| t+T) \, [F(t)+Q(t)P^{-1}(t)]'$$

$$- C(t+T,t)K(t+1)H(t+T)P(t+T)C'(t+T,t) - Q(t) \qquad (43)$$

for $t \geq t_o$ where the initial condition $P(t_o | t_o + T)$ is obtain from the optimal fixed-lag smoothing solution.

Perhaps the most striking feature of the fixed-lag smoothing filter described by Eq. (27) is that it contains two "correction" terms in addition to the "homogeneous" term $F(t) \, \hat{x}(t|t + T)$. This is in contrast to the familiar Kalman-Bucy filter of Eq. (28) which possesses a single "correction" term in addition to the homogeneous term.

The first "correction" term in Eq. (27) is a weighting of the "gain times residual term" $K(t + T)[z(t + T) - H(t + T) \hat{x}(t + T)]$ found in the Kalman-Bucy filter. The function of the smoothing filter gain $C(t + T,t)$ is to weight the "information" in $K(t + T) [z(t + T) - H(t + T) \hat{x}(t + T)]$ and "reflect" it into $\hat{x}(t \quad + T)$.We recall here that the estimate lags the measurement by T units of time.

The second "correction" term, on the other hand, involves a weighting of the difference between the fixed-lag smoothed estimate and the filtered estimate, both at time t. We can view this difference as a smoothing vs. filtering "error" signal.

Now let us recall that $Q(t)$ is the covariance matrix of the system disturbance $w(t)$ in Eq. (1) and that $P(t)$ is the covariance matrix of the filtering error $\tilde{x}(t)$. We note immediately that if there is no system disturbance, then $Q(t) = 0$ for all $t \geq t_o$ and the second "correction" term vanishes. This is plausible for the following reason. If $Q(t) = 0$ for all $t \geq t_o$, the uncertainty in $x(t)$ is due entirely to the uncertainty in $x(t_o)$, the initial state. This uncertainty can only be reduced by examining the measurements $z(t)$ for $t \geq t_o$ in which case the difference $\hat{x}(t|t + T) - \hat{x}(t)$ contains no "new information" not already present in the residual $z - H \hat{x}$.

Now suppose $Q(t) \neq 0$ for $t \geq t_o$. Then,if $\hat{x}(t)$ is an accurate estimate of $x(t)$,

i.e., trace $P(t)$ is small, the weighting factor $Q(t)P^{-1}(t)$ will tend to be "large". This, of course, emphasizes the importance of this second correction term as it should. Indeed, as $P(t) \to 0$, we have $\hat{x}(t) \to x(t)$, and we would expect the correction term to dominate the filter's behavior in an effort to force $\hat{x}(t|t + T)$ into correspondence with $\hat{x}(t)$.

Let us now relate the problem that we have solved here to the classical Wiener smoothing problem(10-13). In the classical formulation, we consider the block diagram shown below in Fig. 2. The message and measurement models are described by Eqs. (1) and (2), respectively, along with the corresponding statistical information which was given in Section 1.0. The ideal smoothing filter is characterized by its $n \times n$ system impulse response matrix which is

$$A_i(t,\tau) = \begin{cases} I\,\delta(t - \tau - T) & t \geq \tau \\ 0 & t < \tau \end{cases}$$

where I is the $n \times n$ identity matrix, δ is the Dirac delta function, and $T > 0$. By the notation $A_i(t,\tau)$, we mean the response or state of the filter at time t for a vector unit impulse (Dirac delta function) input at time τ. From the familiar convolution integral,

$$i(t) = \int_{t_o}^{t} A_i(t,\tau)\, x(\tau)\, d\tau \qquad t \geq t_o + T$$

$$= \int_{t_o}^{t} x(\tau)\, \delta(t - \tau - T)\, d\tau$$

$$= x(t - T)$$

i.e., the ideal state $i(t)$ is the actual state delayed by T units of time.

Although the ideal smoothing filter is physically realizable by a pure delay T, the operation that it is to perform cannot be implemented since we do not have physical access to $x(t)$. Hence, it is the "ideal" against which we compare the performance of the actual smoothing filter which operates on the measurements. We require that this latter filter be linear and that its $n \times m$ impulse response matrix

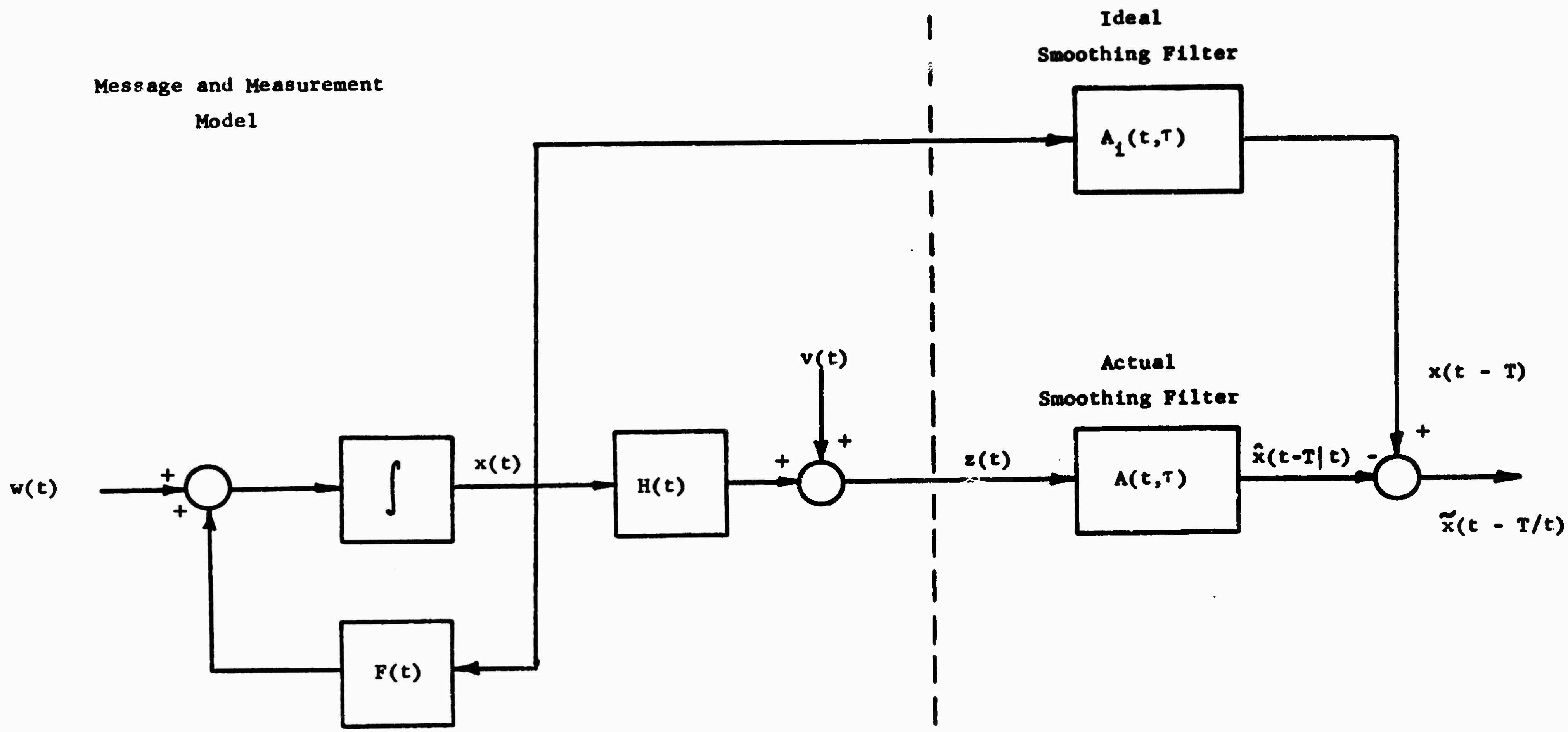


Fig. 2
Block diagram formulation of fixed-lag smoothing filter problem.

$A(t,\tau) = 0$ for all $t < \tau$.

Since the actual smoothing filter requires $\hat{x}(t_o | t_o + T)$ as its initial condition, it cannot begin its operation until measurements over the interval $[t_o, t_o + T]$ have been processed to obtain this initial condition. Hence, in smoothing, we require that $t \geq t_o + T$.

Again, utilizing the convolution integral, we can write

$$\hat{x}(t - T|t) = \int_{t_o}^{t} A(t - T,\tau)\, z(\tau)\, d\tau$$

where $t \geq t_o + T$, or equivalently, by a simple change of variable, we also have

$$\hat{x}(t|t + T) = \int_{t_o}^{t+T} A(t,\tau)\, z(\tau)\, d\tau \tag{44}$$

where now $t \geq t_o$.

The smoothing error is defined as

$$\tilde{x}(t - T|t) = x(t - T) - \hat{x}(t - T|t)$$

for $t \geq t_o + T$, or, equivalently,

$$\tilde{x}(t|t + T) = x(t) - \hat{x}(t|t + T)$$

where $t \geq t_o$.

We then say that a filter of the form described in Eq. (44) that minimizes the mean-square smoothing error

$$S = E[\, \tilde{x}'(t|t + T)\, \tilde{x}(t|t + T)]$$

for all $t \geq t_o$ is an optimal fixed-lag smoothing filter.

In the classical formulation, we see that the problem is that of specifying the filter impulse response matrix $A(t,\tau)$. The results obtained in this paper express the differential equation for the filter rather than its impulse response matrix. However, the latter could be determined from Eqs. (27) and (28) if desired.

In conclusion, we remark that the question of the stability of the optimal

fixed-lag smoothing filter, Eq. (27), its gain matrix, Eq. (37), and the error covariance matrix, Eq. (43), remains as a problem area for future study. We conjecture that if the message and measurement process of Eqs. (1) and (2) is uniformly completely controllable and uniformly completely observable, then the smoothing filter is uniformly asmyptotically stable and the gain and covariance equations have equilibrium solutions for $t_o = -\infty$.

REFERENCES

1. Bryson, A. E. and M. Frazier, "Smoothing for Linear and Nonlinear Dynamic Systems", TDR-63-119, Aero. Sys. Div., Wright-Patterson AFB, Ohio, Feb. 1963, pp. 353-364.

2. Rauch, H. E., F. Tung, and C. T. Striebel, "Maximum Likelihood Estimates of Linear Dynamic Systems", AIAA Jour., vol. 3, no. 8, Aug. 1965, pp. 1445-1450.

3. Rauch, H. E., "Solutions to the Smoothing Problem", IEEE Trans. Auto. Cont., vol. AC-8, no. 4, Oct. 1963, pp. 371-372.

4. Meditch, J. S., "Optimal Fixed-Point Continuous Linear Smoothing", Tech. Rept. 66-108, IPAC Sys. Lab., Dept. of Elec. Eng., Northwestern Univ., Evanston, Ill., Nov. 1966.

5. Kalman, R. E., and R. S. Bucy, "New Results in Linear Filtering and Prediction Theory", Jour. Basic Eng., Trans. ASME, Ser. D, vol. 83, no. 2, March, 1961, pp. 95-108.

6. Kalman, R. E., "New Methods and Results in Linear Prediction and Filtering Theory", Rept. No. 61-1, Research Inst. for Adv. Studies, Martin Co., Baltimore, Md., 1961.

7. Kalman, R. E., "A New Approach to Linear Filtering and Prediction Problems", Jour. Basic Eng., Trans. ASME, Ser. D, vol. 82, no. 2, March 1960, pp. 35-45.

8. Meditch, J. S., "Orthogonal Projection and Discrete Optimal Linear Smoothing", SIAM Jour. on Cont., vol. 5, no. 1, Feb., 1967.

9. Sorenson, H. W., "Kalman Filtering Techniques", in "Advances in Control Systems - Theory and Applications", Vol. 3, C. T. Leondes, ed., Academic Press, New York, 1966, pp. 219-292.

10. Wiener, N., "Extrapolation, Interpolation, and Smoothing of Stationary Time Serier", John Wiley and Sons, Inc., New York, 1957, (Third printing).

11. Lee, Y. W., "Statistical Theory of Communication", John Wiley and Sons, Inc., New York, 1960.

12. Newton, G. C., Jr., L. A. Gould, and J. F. Kaiser, "Analytical Design of Linear Feedback Controls", John Wiley and Sons, Inc., New York, 1957.

13. Laning, J. H., and R. H. Battin, "Random Processes in Automatic Control", McGraw-Hill Book Co., Inc., New York, 1956.

Unclassified

Security Classification

DOCUMENT CONTROL DATA - R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Information-Processing And Control SYSTEMS LABORATORY Northwestern University, Evanston, Illinois 60201	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b GROUP Not applicable

3. REPORT TITLE

ON OPTIMAL LINEAR SMOOTHING THEORY

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Not applicable

5. AUTHOR(S) (Last name, first name, initial)

Meditch, James S.

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
March 1967	27	22

8a. CONTRACT XXXXXXXX NO. N00014-66-C0020-A03 b. PROJECT NO NR 373-502/3-14-66 Electronics Branch c. Not applicable d. Not applicable	9a. ORIGINATOR'S REPORT NUMBER(S) 67-105 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report) Not applicable

10. AVAILABILITY/LIMITATION NOTICES

Qualified requesters may obtain copies of this report from the Defense Documentation Center.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
Not applicable	Office of Naval Research, Washington, D.C.

13. ABSTRACT

The algorithm for generating the smoothed estimate $\hat{x}(t/t + T)$ of the state $x(t)$ of a continuous linear system, where t is continuous time, T is a positive real constant, and $t + T$ is the time of the most recent measurement, is developed. A linear matrix differential equation whose solution gives the covariance matrix of the smoothing error $\tilde{x}(t/t + T)=\hat{x}(t/t + T)$ is then derived. Computational aspects involved in mechanizing the algorithm are discussed in terms of the algorithm's dependence on the solution of the prediction, filtering, and fixed-point smoothing problems. The results are then discussed in terms of the classical Wiener smoothing problem.

Unclassified

Security Classification

14. KEY WORDS	LINK A ROLE	LINK A WT	LINK B ROLE	LINK B WT	LINK C ROLE	LINK C WT
Smoothing theory						
Linear estimation						
Optimal linear smoothing						
Filtering theory						
Fixed point smoothing						
Data smoothing algorithm						

INSTRUCTIONS

1. ORIGINATING ACTIVITY: Enter the name and address of the contractor, subcontractor, grantee, Department of Defense activity or other organization (*corporate author*) issuing the report.

2a. REPORT SECURITY CLASSIFICATION: Enter the overall security classification of the report. Indicate whether "Restricted Data" is included. Marking is to be in accordance with appropriate security regulations.

2b. GROUP: Automatic downgrading is specified in DoD Directive 5200.10 and Armed Forces Industrial Manual. Enter the group number. Also, when applicable, show that optional markings have been used for Group 3 and Group 4 as authorized.

3. REPORT TITLE: Enter the complete report title in all capital letters. Titles in all cases should be unclassified. If a meaningful title cannot be selected without classification, show title classification in all capitals in parenthesis immediately following the title.

4. DESCRIPTIVE NOTES: If appropriate, enter the type of report, e.g., interim, progress, summary, annual, or final. Give the inclusive dates when a specific reporting period is covered.

5. AUTHOR(S): Enter the name(s) of author(s) as shown on or in the report. Enter last name, first name, middle initial. If military, show rank and branch of service. The name of the principal author is an absolute minimum requirement.

6. REPORT DATE: Enter the date of the report as day, month, year, or month, year. If more than one date appears on the report, use date of publication.

7a. TOTAL NUMBER OF PAGES: The total page count should follow normal pagination procedures, i.e., enter the number of pages containing information.

7b. NUMBER OF REFERENCES: Enter the total number of references cited in the report.

8a. CONTRACT OR GRANT NUMBER: If appropriate, enter the applicable number of the contract or grant under which the report was written.

8b, 8c, & 8d. PROJECT NUMBER: Enter the appropriate military department identification, such as project number, subproject number, system numbers, task number, etc.

9a. ORIGINATOR'S REPORT NUMBER(S): Enter the official report number by which the document will be identified and controlled by the originating activity. This number must be unique to this report.

9b. OTHER REPORT NUMBER(S): If the report has been assigned any other report numbers (*either by the originator or by the sponsor*), also enter this number(s).

10. AVAILABILITY/LIMITATION NOTICES: Enter any limitations on further dissemination of the report, other than those imposed by security classification, using standard statements such as:

(1) "Qualified requesters may obtain copies of this report from DDC."

(2) "Foreign announcement and dissemination of this report by DDC is not authorized."

(3) "U. S. Government agencies may obtain copies of this report directly from DDC. Other qualified DDC users shall request through ________________."

(4) "U. S. military agencies may obtain copies of this report directly from DDC. Other qualified users shall request through ________________."

(5) "All distribution of this report is controlled. Qualified DDC users shall request through ________________."

If the report has been furnished to the Office of Technical Services, Department of Commerce, for sale to the public, indicate this fact and enter the price, if known.

11. SUPPLEMENTARY NOTES: Use for additional explanatory notes.

12. SPONSORING MILITARY ACTIVITY: Enter the name of the departmental project office or laboratory sponsoring (*paying for*) the research and development. Include address.

13. ABSTRACT: Enter an abstract giving a brief and factual summary of the document indicative of the report, even though it may also appear elsewhere in the body of the technical report. If additional space is required, a continuation sheet shall be attached.

It is highly desirable that the abstract of classified reports be unclassified. Each paragraph of the abstract shall end with an indication of the military security classification of the information in the paragraph, represented as *(TS)*, *(S)*, *(C)*, or *(U)*.

There is no limitation on the length of the abstract. However, the suggested length is from 150 to 225 words.

14. KEY WORDS: Key words are technically meaningful terms or short phrases that characterize a report and may be used as index entries for cataloging the report. Key words must be selected so that no security classification is required. Identifiers, such as equipment model designation, trade name, military project code name, geographic location, may be used as key words but will be followed by an indication of technical context. The assignment of links, roles, and weights is optional.